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Heat kernel expansion and energy–momentum tensor for fermion field at finite temperature in curved spacetime

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Abstract. The method of heat kernel expansion at finite temperature in curved space is proposed. We consider the case of a second-order operator $A(x, \nabla_\alpha) = -\nabla^\alpha \nabla_\alpha + X$. The thermal DeWitt–Seeley–Gilkey coefficients $E_m(x, \beta_0, \tau)$ $(m = 0, 2, 4)$ are derived. The effective action and the energy–momentum tensor for free fermion gas at finite temperature in curved space are obtained.

1. Introduction

The heat kernel expansion, the effective action and the energy–momentum tensor at finite temperature in curved spacetime are important topics in thermal quantum field theory [TQFT]. They are useful in background problems. There exist two different ways which have been developed in studying GFT and TQFT in curved spacetime. One of them is the usual quantum field method, originated by DeWitt [1] which is often used by physicists [2, 3]. The other is the geometric method, originated from Seeley [4] and Gilkey [5] which is often used by mathematicians [6].

Many people have considered the boson field at finite temperature in various special curved spacetime [9–13]. Only a few authors [14, 15] have dealt with the more general static spacetime with $g_{0\alpha} = 0$ but g_{00} not constant.

In this article we examine the effective action and energy–momentum tensor for a fermion field at finite temperature in static curved spacetime by heat kernel expansion and find the so-called anomalous terms which exist both for the boson field [14] as well as the fermion field.

2. Review of Gusynin's algorithm [6]

Let $A(x, \nabla_\alpha)$ be a second or higher-order differential operator, working in d dimensional Euclidean space, x is the coordinate. Formally we have the operator transformation

$$e^{-\tau A} = \frac{1}{2\pi i} \int_c d\lambda\, e^{-\tau\lambda} \frac{1}{\lambda - A} = \frac{i}{2\pi} \int_c d\lambda\, e^{-\tau\lambda} \frac{1}{A - \lambda} \tag{2.1}$$

where c is the contour in the λ plane which encloses all eigenvalues of the operator A. Note that the parameter τ is not a time coordinate, it is only an evolution parameter which sometimes is called the $(d + 1)$th fictitious time coordinate.

The heat kernel $H = H(x, x', \tau, A)$ and the modified Green function $G = G(x, x', \lambda, A)$ related to A are defined as

$$\begin{aligned} H(x, x', \tau, A) &\equiv \langle x'|e^{-\tau A(x,\nabla_\alpha)}|x\rangle \\ &= \frac{i}{2\pi}\int_c d\lambda\, e^{-\tau\lambda}\langle x'|\frac{1}{A-\lambda}|x\rangle = \frac{i}{2\pi}\int_c d\lambda\, e^{-\tau\lambda} G(x, x', \lambda, A) \end{aligned} \tag{2.2}$$

where

$$G(x, x', \lambda, A) \equiv \langle x'|\frac{1}{A-\lambda}|x\rangle$$

or

$$[A(x, \nabla_\alpha) - \lambda]G(x, x', \lambda, A) = \langle x'|x\rangle. \tag{2.3}$$

It is obvious that $H(x, x', \tau, A)$ satisfies the heat equation

$$\left[\frac{\partial}{\partial\tau} + A(x, \nabla_\alpha)\right] H(x, x', \tau, A) = 0. \tag{2.4}$$

One of the main points of Gusynin's work is that the modified Green function in curved space at zero temperature can be expanded into a generalized Fourier integral as follows

$$G(x, x', \lambda, A) \equiv \langle x'|\frac{1}{A-\lambda}|x\rangle = \frac{1}{(2\pi)^d}\int \frac{d^d k}{\sqrt{|g(x)|}} e^{il(x,x',k)}\sigma(x, x', k, \lambda) \tag{2.5}$$

where $l(x, x', k)$ is called phase and $\sigma(x, x', k, \lambda)$ is called amplitude. The argument A in l and σ is omitted. $l(x, x', k)$ is a real biscalar function under general coordinate transformations and $l(x, x', k)|_{x=x'} = 0$. For flat space $l(x, x')$ may be reduced to

$$l(x, x', k) = k_\alpha(x - x')^\alpha \tag{2.6}$$

$$\frac{\partial}{\partial x^\alpha} l(x, x', k) = k_\alpha \qquad \frac{\partial}{\partial k_\alpha} l(x, x', k) = (x - x')^\alpha. \tag{2.7}$$

Next we require $l(x, x', k)$ to satisfy the initial conditions

$$[\![\{\nabla_{\alpha_1}\nabla_{\alpha_2}\ldots\nabla_{\alpha_m} l(x, x', k)\}]\!] = \{\nabla_{\alpha_1}\nabla_{\alpha_2}\ldots\nabla_{\alpha_m} l(x, x', k)\}|_{x=x'} = \begin{cases} k_\alpha & m = 1 \\ 0 & m \neq 1. \end{cases} \tag{2.8}$$

The symbol $\{\}$ denotes symmetrization in all indices, $[\![\,]\!]$ denotes to take the coincidence limit. For detailed calculation we want to use a non-symmetrized covariant derivative of the form

$$[\![\nabla_{\alpha_1}\nabla_{\alpha_2}\ldots\nabla_{\alpha_m} l(x, x', k)]\!] \equiv [\![\nabla_{\alpha_1\alpha_2\ldots\alpha_m} l(x, x', k)]\!]. \tag{2.9}$$

Such quantities are obtained directly from (2.8) by reducing all terms to a certain fixed indices order. In such a process we must use the commutation relation of covariant derivatives

$$[\nabla_\alpha, \nabla_\beta]A_{\alpha_1,\alpha_2\ldots\alpha_n} = -R^\lambda{}_{\alpha_i\alpha\beta}A_{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_n} + T^\lambda{}_{\alpha\beta}\nabla_\lambda A_{\alpha_1,\ldots\alpha_n} \tag{2.10}$$

where the Riemannian and torsion tensors are defined as follows:

$$R^{\lambda}{}_{\tau\alpha\beta} = \partial_\alpha \Gamma^{\lambda}{}_{\tau\beta} - \partial_\beta \Gamma^{\lambda}{}_{\tau\alpha} + \Gamma^{\lambda}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\tau\beta} - \Gamma^{\lambda}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\tau\alpha} \tag{2.11}$$

$$T^{\lambda}{}_{\alpha\beta} = \Gamma^{\lambda}{}_{\alpha\beta} - \Gamma^{\lambda}{}_{\beta\alpha} \tag{2.12}$$

and $\Gamma^{\lambda}{}_{\alpha\beta}$ is the affine connection. In what follows we shall consider mainly the case of a space manifold without torsion, but our algorithm is directly applicable to the space manifold with torsion.

From (2.8) and (2.10) it is easy to find the coincidence limit of the lower-order covariant derivatives of $l(x, x', k)$

$$[\![l]\!] = 0 \qquad [\![\nabla_\alpha l]\!] = k^\alpha \tag{2.13a}$$

$$[\![(\nabla_{\alpha\beta} + \nabla_{\beta\alpha})l]\!] = 2[\![\nabla_{\alpha\beta} l]\!] = 0 \tag{2.13b}$$

$$[\![\{\nabla_{\alpha\beta\lambda} l\}]\!] = 2[\![(\nabla_{\alpha\beta\lambda} + \nabla_{\beta\lambda\alpha} + \nabla_{\lambda\alpha\beta})]\!] = 2[\![3\nabla_{\alpha\beta\lambda} l - R^{\tau}{}_{\lambda\alpha\beta}\nabla_\tau l - R^{\tau}{}_{\beta\lambda\alpha}\nabla_\tau l]\!] = 0$$

then we have

$$[\![\nabla_{\alpha\beta\lambda} l]\!] = -\tfrac{2}{3} k_\tau S^{\tau}{}_{\lambda\alpha\beta} \qquad S^{\tau}{}_{\lambda\alpha\beta} = \tfrac{1}{2}(R^{\tau}{}_{\lambda\alpha\beta} + R^{\tau}{}_{\beta\alpha\lambda}). \tag{2.13c}$$

The second main point of Gusynin's work may be recast as follows, though he did not express it explicitly. As is well known the Fourier integral expansion of the delta function in flat space is

$$\langle x'|x\rangle = \frac{1}{(2\pi)^d}\int \mathrm{d}^d\, k \mathrm{e}^{\mathrm{i}k_\alpha (x-x')^\alpha}. \tag{2.14}$$

In curved space with a fibre bundle structure we revise (2.14) to

$$\langle x'|x\rangle = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}} \mathrm{e}^{\mathrm{i}l(x,x',k)} I(x, x') \tag{2.15}$$

where $I(x, x')$ is a biscalar function in the base space sector and carries two bundle indices, which to some extent, is analogous to the displacement matrix in DeWitt's method [1]. $I(x, x')$ satisfies the initial conditions

$$[\![I(x, x')]\!] = I(x) \tag{2.16}$$

$$[\![\{\nabla_{\alpha_1}\nabla_{\alpha_2}\ldots\nabla_{\alpha_m}\} I(x, x')]\!] = 0 \qquad \text{when } m \geqslant 1. \tag{2.17}$$

Similarly to the case of $l(x, x', k)$ we need the commutation relation of the covariant derivatives of the tensor field in bundle space

$$[\nabla_\alpha, \nabla_\beta] S_{\alpha_1\alpha_2\ldots\alpha_n} = -R^{\lambda}{}_{\alpha_i\alpha\beta} S_{\alpha_1\alpha_2\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_n} + T^{\lambda}{}_{\alpha\beta}\nabla_\lambda S_{\alpha_1\alpha_2\ldots\alpha_n} + W_{\alpha\beta} S_{\alpha_1\alpha_2\ldots\alpha_n} \tag{2.18}$$

where $W_{\alpha\beta}$ is the bundle curvature

$$W_{\alpha\beta} = \partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha + [\omega_\alpha, \omega_\beta] \tag{2.19}$$

and ω_α is the connection of the bundle. We shall consider only the Riemannian manifold without torsion, $T^\lambda{}_{\alpha\beta} = 0$.

From (2.17) when $m = 1$, it is obvious that

$$[\![\nabla_\alpha I(x, x')]\!] = 0 \tag{2.20a}$$

When $m = 2$, we have

$$\begin{aligned}[\![\{\nabla_\alpha \nabla_\beta\} I(x, x')]\!] &= [\![\nabla_\alpha \nabla_\beta I + \nabla_\beta \nabla_\alpha I]\!] \\ &= [\![2\nabla_\alpha \nabla_\beta I + T^\lambda{}_{\alpha\beta} \nabla_\lambda I + W_{\beta\alpha} I]\!] \\ &= 2\nabla_\alpha \nabla_\beta I + W_{\beta\alpha} = 0\end{aligned}$$

then

$$[\![\nabla_\alpha \nabla_\beta I]\!] \equiv I_{\alpha\beta} = \tfrac{1}{2} W_{\alpha\beta}. \tag{2.20b}$$

From (2.5) we obtain

$$\begin{aligned}(A - \lambda)G = \langle x'|x\rangle &= \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}} \{A(x, \nabla_\alpha) - \lambda\}\{\mathrm{e}^{\mathrm{i}l(x,x',k)} \sigma(x, x', k, \lambda)\} \\ &= \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}} \mathrm{e}^{\mathrm{i}l(x,x',K)} \{\hat{A}(x, \mathrm{i}(\nabla_\alpha l) + \nabla_\alpha) - \lambda\} \sigma(x, x', k, \lambda).\end{aligned} \tag{2.21}$$

Combining (2.21) with (2.15), it follows that

$$\{\hat{A}(x, \mathrm{i}(\nabla_\alpha l) + \nabla_\alpha) - \lambda\} \sigma(x, x', k, \lambda) = I(x, x'). \tag{2.22}$$

It must be stressed that one cannot obtain (2.22) without the help of (2.15).

Let $A(x, \nabla_\alpha)$ be a second-order differential operator, for instance

$$A(x, \nabla_\alpha) = -\Box + X = -\nabla^\alpha \nabla_\alpha + X \tag{2.23}$$

where X is not an operator, then

$$\hat{A}(x, \mathrm{i}(\nabla_\alpha l) + \nabla_\alpha) = (\nabla^\alpha l)(\nabla_\alpha l) - \mathrm{i}\Box l - 2\mathrm{i}(\nabla^\alpha l)\nabla_\alpha - \Box + X \tag{2.24}$$

and (2.22) becomes

$$\{(\nabla^\alpha l)(\nabla_\alpha l) - \mathrm{i}\Box l - 2\mathrm{i}(\nabla^\alpha l)\nabla_\alpha - \Box + X - \lambda\} \sigma(x, x', k, \lambda) = I(x, x'). \tag{2.25}$$

Next we set

$$\sigma(x, x', k, \lambda) = \sum_{m=0}^{\infty} \varepsilon^{2+m} \sigma_m(x, x', k, \lambda) \qquad 0 < \varepsilon < 1 \tag{2.26}$$

and rescale l and λ as

$$l \to l' = \varepsilon^{-1} l \qquad \lambda \to \lambda' = \varepsilon^{-2} \lambda. \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.25) we have

$$\{\varepsilon^{-2}(\nabla^\alpha l)(\nabla_\alpha l) - \varepsilon^{-2}\lambda + \varepsilon^{-1}\mathrm{i}(\Box l) - \varepsilon^{-1}2\mathrm{i}(\nabla^\alpha l)\nabla_\alpha - \Box + X\}\sum_{m=0}^{\infty}\varepsilon^{2+m}\sigma_m(x,x',k,\lambda)$$
$$= I(x,x'). \tag{2.28}$$

Equating the coefficients of ε^m on both sides of (2.28), we obtain the recursion relations

$$\{(\nabla^\alpha l)(\nabla_\alpha l) - \lambda\}\sigma_0 = I(x,x') \tag{2.29}$$

$$\{(\nabla^\alpha l)(\nabla_\alpha l) - \lambda\}\sigma_1 - \mathrm{i}\{\Box l + 2(\nabla^\alpha l)\nabla_\alpha\}\sigma_0 = 0 \tag{2.30}$$

$$\{(\nabla^\alpha l)(\nabla_\alpha l) - \lambda\}\sigma_m - \mathrm{i}\{\Box l + 2(\nabla^\alpha l)\nabla_\alpha\}\sigma_{m-1} - (\Box - X)\sigma_{m-2} = 0 \qquad m \geqslant 2. \tag{2.31}$$

Solving (2.29) we obtain

$$\sigma_0 = \frac{1}{(\nabla_\alpha l)(\nabla^\alpha l) - \lambda} I \equiv bI \qquad b = \frac{1}{(\nabla_\alpha l)(\nabla^\alpha l) - \lambda}. \tag{2.32}$$

Since

$$[\![(\nabla_\alpha l)(\nabla^\alpha l)]\!] = k_\alpha k^\alpha = k^2$$

then

$$[\![b]\!] = \frac{1}{k^2 - \lambda} \equiv B \qquad [\![\sigma_0]\!] = BI. \tag{2.33}$$

Solving (2.30) we have

$$\sigma_1 = \mathrm{i}b\{(\Box l)\sigma_0 + 2(\nabla^\alpha l)(\nabla_\alpha \sigma_0)\}. \tag{2.34}$$

Since

$$[\![\Box l]\!] = [\![\nabla^\alpha \nabla_\alpha l]\!] = 0$$

then

$$[\![\sigma_1]\!] = 2\mathrm{i}Bk^\alpha [\![\nabla_\alpha \sigma_0]\!] = 0. \tag{2.35}$$

Similarly we derive

$$[\![\nabla_{\alpha\beta}\sigma_0]\!] = -2\mathrm{i}B^2 k^\lambda k^\tau l_{\alpha\beta\lambda\tau} + BI. \tag{2.36}$$

$$[\![\sigma_2]\!] = -2B^3 k^\alpha k^\beta (l^\lambda{}_{\lambda\alpha\beta} + l^\lambda{}_{\alpha\lambda\beta}) - B^2 X. \tag{2.37}$$

In the above expressions, we have introduced the symbols

$$[\![\nabla_{\alpha\beta\ldots\lambda} l(x,x',k)]\!] \equiv k^\tau l_{\alpha\beta\ldots\lambda\tau} \tag{2.38}$$

$$[\![\nabla_{\alpha\beta\ldots\lambda} I(x,x')]\!] \equiv I_{\alpha\beta\ldots\lambda}. \tag{2.39}$$

3. The heat kernel expansion at finite temperature

The zero temperature formalism can be extended to finite temperature. We assume the curved spacetime is static or quasi-static and the metric can be written as follows:

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta = g_{00}\,dt^2 + g_{ij}\,dx^i\,dx^j \qquad i,j = 1,2,\ldots,(d-1). \tag{3.1}$$

g_{00} and g_{ij} are independent of time t for the static case, and for the quasi-static case, they vary very slowly with time t. We only consider the static case in this paper.

In TQFT one considers the ensemble averages of the operator $\hat{o}$. For the canonical ensemble, we have

$$\langle\hat{o}\rangle_{\beta_0} = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta_0\hat{H}}\hat{o})}{\mathrm{Tr}(\mathrm{e}^{-\beta_0\hat{H}})} \tag{3.2}$$

where $\hat{H}$ is the second quantized field Hamiltonian and the trace is taken over the quantum field Hilbert space.

It can be shown that [8] the Green function or propagator at finite temperature is periodic (antiperiodic)

$$G_{\beta_0}(x,x') \equiv G_{\beta_0}(\boldsymbol{x},t,\boldsymbol{x}',t') = \pm G_{\beta_0}(\boldsymbol{x},t+\beta_0,\boldsymbol{x}',t') = \pm G_{\beta_0}(\boldsymbol{x},t,\boldsymbol{x}',t'+\beta_0). \tag{3.3}$$

The periodic (antiperiodic) property of the thermal Green function in the t variable for boson (fermion) fields is usually interpreted as broken spacetime symmetry. One must remember that t is not a real physical variable in the sense that time is, it is only a formal device for introducing thermal properties.

At finite temperature, we revise equation (2.15) for the fermion field as

$$\langle x'|x\rangle_{\beta_0}^{(f)} = \frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{\sqrt{|g(x)|}}\mathrm{e}^{\mathrm{i}l(x,x',k)}\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right]I(x,x') \tag{3.4}$$

and we propose that the generalized Fourier transformation of the thermal modified Green function for the fermion field is

$$\begin{aligned}G_{\beta_0}^{(f)}(x,x',\lambda) &= \langle x'|\frac{1}{A-\lambda}|x\rangle_{\beta_0}^{(f)}\\ &= \frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{\sqrt{|g(x)|}}\mathrm{e}^{\mathrm{i}l(x,x',k)}\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right]\sigma(x,x',k,\lambda).\end{aligned} \tag{3.5}$$

The reason that (3.5) holds is that $1/(A(x,\nabla_\alpha)-\lambda)$ is a differential operator, and not a quantum field operator.

From (3.5) we obtain

$$\begin{aligned}(A-\lambda)G_{\beta_0}^{(f)}(x,x',\lambda) &= \frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{\sqrt{|g(x)|}}\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right]\\ &\quad\times\{A(x,\nabla_\alpha)-\lambda\}\{\mathrm{e}^{\mathrm{i}l(x,x',k)}\sigma(x,x',k,\lambda)\}\\ &= \frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{\sqrt{|g(x)|}}\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right]\\ &\quad\times\mathrm{e}^{\mathrm{i}l(x,x',k)}\{\hat{A}(x,\mathrm{i}(\nabla_\alpha l)+\nabla_\alpha)-\lambda\}\sigma(x,x',k,\lambda).\end{aligned} \tag{3.6}$$

Combining (3.6) with (3.4), one arrives at

$$\{\hat{A}(x, \mathrm{i}(\nabla_\alpha l) + \nabla_\alpha) - \lambda\}\sigma(x, x', k, \lambda) = I(x, x') \tag{3.7}$$

which is the same as (2.22), derived in the zero temperature case.

From (3.5) it follows that the heat kernel expansion for the fermion field at finite temperature is

$$\begin{aligned}\langle x'|\mathrm{e}^{-\tau A}|x\rangle^{(f)}_{\beta_0} &= \langle x'|\frac{\mathrm{i}}{2\pi}\int_c \mathrm{d}\lambda\, \mathrm{e}^{-\tau\lambda}\frac{1}{A-\lambda}|x\rangle^{(f)}_{\beta_0} \\ &= \frac{\mathrm{i}}{2\pi}\int_c \mathrm{d}\lambda\, \mathrm{e}^{-\tau\lambda}\langle x'|\frac{1}{A-\lambda}|x\rangle^{(f)}_{\beta_0} \\ &= \frac{\mathrm{i}}{2\pi}\int_c \mathrm{d}\lambda\, \mathrm{e}^{-\tau\lambda}G^{(f)}_{\beta_0}(x, x', \lambda) \\ &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}}\int_c \frac{\mathrm{i}\,\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\tau\lambda}\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0 - \frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right] \\ &\quad \times \mathrm{e}^{\mathrm{i}l(x,x',k)}\sigma(x, x', k, \lambda).\end{aligned} \tag{3.8}$$

Setting

$$\sigma(x, x', k, \lambda) = \sum_{m=0}^{\infty}\varepsilon^m \sigma_m(x, x', k, \lambda) \qquad 0 < \varepsilon < 1 \tag{3.9}$$

and

$$[\![\langle x'|\mathrm{e}^{-\tau A}|x\rangle^{(f)}_{\beta_0}]\!] = \sum_{m=0}^{\infty}\varepsilon^m H^{(f)}_m(x, \beta_0, \tau) \tag{3.10}$$

and equating the coefficients of ε^m on both sides of (3.8), we derive

$$\begin{aligned}H^{(f)}_m(x, \beta_0, \tau) &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}}\int_c \frac{\mathrm{i}\,\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\tau\lambda} \\ &\quad \times \sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[k_0 - \frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right][\![\sigma_m]\!](x, k, \lambda).\end{aligned} \tag{3.11}$$

Note that $[\sigma_m](x, k, \lambda)$ is proportional to $k_{\alpha_1}k_{\alpha_2}\ldots k_{\alpha_m}$. It is obvious that $H^{(f)}_m(x, \beta_0, \tau) = 0$ when m is odd and $H^{(f)}_m(x, \beta_0, \tau) \neq 0$ when m is even. If we change the scale of k and λ as

$$k \to k' = \tau^{-1/2}k \qquad \lambda \to \lambda' = \tau^{-1}\lambda \tag{3.12}$$

it is easy to show

$$[\![\sigma_m]\!](x, \tau^{-1/2}k, \tau^{-1}\lambda) = \tau^{(m+2)/2}[\![\sigma_m]\!](x, k, \lambda) \tag{3.13}$$

then (3.11) becomes

$$H^{(f)}_m(x, \beta_0, \tau) = \tau^{(m-d)/2}E^{(f)}_m(x, \beta_0, \tau) \tag{3.14}$$

then (3.11) becomes

$$H_m^{(f)}(x,\beta_0,\tau)=\tau^{(m-d)/2}E_m^{(f)}(x,\beta_0,\tau) \tag{3.14}$$

where

$$E_m^{(f)}(x,\beta_0,\tau)=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{\sqrt{|g(x)|}}\int_c\frac{\mathrm{i}\,\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\lambda}\times\sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[\frac{k_0}{\sqrt{\tau}}-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right][\![\sigma_m]\!](x,k,\lambda) \tag{3.15}$$

are called the thermal DeWitt–Seeley–Gilkey coefficients.

From (3.15) we derive the thermal DeWitt–Seeley–Gilkey coefficients $E_m^{(f)}(x,\beta_0,\tau)$, $m=0,2,4$ for the fermion field as follows (appendix A):

$$E_0^{(f)}=\frac{1}{(4\pi)^{d/2}}\left[1+2\sum_{n=1}^{\infty}(-1)^n\mathrm{e}^{-n^2\beta^2/4\tau}\right]=\frac{1}{(4\pi)^{d/2}}\theta_4(z|\hat{\tau}) \tag{3.16}$$

$$E_2^{(f)}=\frac{1}{(4\pi)^{d/2}}\left\{\left(\frac{1}{6}R-X\right)\theta_4(z|\hat{\tau})-\frac{2}{3}R_0^0\sum_{n=1}^{\infty}(-1)^n\frac{n^2\beta^2}{4\tau}\mathrm{e}^{-n^2\beta^2/4\tau}\right\} \tag{3.17}$$

where

$$\beta=\sqrt{g_{00}}\beta_0\qquad\beta_0=\text{constant}$$
$$\theta_4(z|\hat{\tau})=1+2\sum_{n=1}^{\infty}(-1)^n\mathrm{e}^{-n^2\beta^2/4\tau} \tag{3.18}$$

and $\theta_4(z|\hat{\tau})$ is the theta function, $\hat{\tau}=\mathrm{i}\beta^2/4\pi\tau$ is the period, $z=2\pi j$, j is a non-negative integer.

When $\beta_0=(1/T_0)\to\infty$ the temperature $T_0=(1/\beta_0)\to 0$, and equation (3.17) be reduced to

$$E_2(x)=\frac{1}{(4\pi)^{d/2}}\left[\frac{1}{6}R-X\right] \tag{3.19}$$

where $E_2(x)$ is the zero temperature DeWitt–Seeley–Gilkey coefficient. The second term of (3.17) is an anomalous term which approaches zero when $\beta_0\to 0$ and $\beta_0\to\infty$ or $T_0\to\infty$ and $T_0\to 0$. It is finite if $T_0\neq 0$ and $T_0\neq\infty$ and there is no 'zero temperature partner' of it.

Similarly we obtain $E_2^{(b)}(x,\beta,\tau)$ for the boson field

$$E_2^{(b)}(x,\beta,\tau)=\frac{1}{(4\pi)^{d/2}}\left\{\left(\frac{1}{6}R-X\right)\left[1+2\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\beta^2/4\tau}\right]-\frac{2}{3}R_0^0\sum_{n=1}^{\infty}\frac{n^2\beta^2}{4\tau}\mathrm{e}^{-n^2\beta^2/4\tau}\right\}$$
$$=\frac{1}{(4\pi)^{d/2}}\left\{\left(\frac{1}{6}R-X\right)\theta_3(z|\hat{\tau})-\frac{2}{3}R_0^0\sum_{n=1}^{\infty}\frac{n^2\beta^2}{4\tau}\mathrm{e}^{-n^2\beta^2/4\tau}\right\}. \tag{3.20}$$

The difference between $E_2^{(f)}(x, \beta_0, \tau)$ and $E_2^{(b)}(x, \beta_0, \tau)$ is that one replaces $\sum_{n=1}^{\infty}(-1)^n$ in the fermion field by $\sum_{n=1}^{\infty}$ in the boson field. Note that $E_2^{(b)}(x, \beta_0, \tau)$ also contains an anomalous term.

By using the formulas (3.15), (A3), (A5), (B2), and after a lengthy calculation, we obtain the thermal $E_4^{(f)}(x, \beta_0, \tau)$ coefficient for the fermion field as

$$E_4^{(f)}(x, \beta_0, \tau) = \frac{1}{(4\pi)^{d/2}}\left\{\left[\frac{1}{180}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{72}R^2 + \frac{1}{30}\Box R - \frac{1}{6}XR + \frac{1}{2}X^2 - \frac{1}{6}\Box X + \frac{1}{12}W_{\alpha\beta}W^{\alpha\beta}\right]\theta_4(z|\hat{\tau}) + A_4^{(f)}(x, \beta_0, \tau)\right\} \tag{3.21}$$

where $A_4^{(f)}(x, \beta_0, \tau)$ denotes the anomalous term which is very complicated.

4. The effective action and energy–momentum tensor

The effective action for the fermion field at finite temperature in four-dimensional spacetime is [2]

$$\begin{aligned}\ln Z_{\beta_0}^{(f)} &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g(x)|}\alpha_f^{(1)}(x, \beta_0)\\ &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g(x)|}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\langle x|\mathrm{e}^{-\tau A(x,\nabla_a)}|x\rangle_{\beta_0}^{(f)}\\ &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g(x)|}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\sum_{m=0}^{\infty}\tau^{(m-4)/2}E_m^{(f)}(x, \beta_0, \tau)\\ &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g(x)|}\int_0^{\infty}\mathrm{d}\tau\{[\tau^{-3}E_0(x) + \tau^{-2}E_2(x) + \tau^{-1}E_4(x) + \ldots]\theta_4(z|\hat{\tau})\\ &\quad + \text{anomalous terms}\}.\end{aligned} \tag{4.1}$$

Note that we choose the heat kernel expansion of type I in (4.1) (appendix C). After some straightforward calculation, equation (4.1) is reduced to

$$\ln Z_{\beta_0}^{(f)} = \int \mathrm{d}^4x\sqrt{|g(x)|}\left\{Y_f(4)\frac{1}{\pi^2}\zeta(4)\beta^{-4} + Y_f(2)\frac{1}{4\pi^2}\zeta(2)\beta^{-2}\left[\frac{1}{6}R - X\right] + \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + \gamma + 2\ln 2\right]E_4(x) + \cdots\right\} \tag{4.2}$$

where $\gamma = 0.57221$ is the Euler constant and

$$Y_f(s) = 1 - 2^{1-s} \qquad \text{for } s = 2, 4, \ldots . \tag{4.3a}$$

For the boson field one replaces $Y_f(s)$ in (4.2) by $Y_b(s)$

$$Y_b(s) = 1 \qquad \text{for } s = 2, 4, \ldots . \tag{4.3b}$$

If one uses the heat kernel expansion of type II, one obtains the effective action for the fermion field at finite temperature:

$$\ln Z_{\beta_0}^{(f)} = \int d^4x \sqrt{|g(x)|} \times \sum_{n=1}^{\infty}(-1)^n \left\{ \frac{1}{(4\pi)^2}\left[\int_0^\infty \frac{d\tau}{\tau^3} e^{-(\tau X + n^2\beta^2/4\tau)} + \frac{1}{6}R\int_0^\infty \frac{d\tau}{\tau^2} e^{-(\tau X+n^2\beta^2/4\tau)}\right] + F_4(x)\int_0^\infty \frac{d\tau}{\tau} e^{-(\tau X+n^2\beta^2/4\tau)} + \cdots \right\}$$

$$= \int d^4x \sqrt{|g(x)|}\sum_{n=1}^{\infty}(-1)^n \frac{1}{(4\pi)^2}\left\{2\left(\frac{4X}{n^2\beta^2}\right)k_2(\sqrt{X}n\beta) + \frac{1}{3}R\left(\frac{4X}{n^2\beta^2}\right)^{1/2} k_1(\sqrt{X}n\beta) + \cdots\right\}$$

$$= \int d^4x\sqrt{|g(x)|}\left\{\frac{1}{\pi^2}Y_f(4)\zeta(4)\beta^{-4} - \frac{1}{4\pi^2}Y_f(2)\zeta(2)\beta^{-2}X - \frac{1}{32\pi^2}\left(\gamma - \frac{3}{4} + \frac{1}{2}\ln\frac{1}{4}X\beta^2\right)X^2 + \frac{1}{24\pi^2}Y_f(2)\zeta(2)\beta^{-2}R + \frac{1}{48\pi^2}\left(\gamma - \frac{1}{2} + \frac{1}{2}\ln\frac{1}{4}X\beta^2\right)XR + \cdots\right\} \tag{4.4}$$

where $k_\nu(z)$ is the modified Bessel function of the third kind.

The contribution of the anomalous term in (C8) to the effective action is

$$\ln Z_{\beta_0}^{(f)a} = \frac{1}{32\pi^2}\int d^4x \sqrt{|g(x)|}\int_0^\infty \frac{d\tau}{\tau^2}\left(-\frac{2}{3}R_0^0\right)\sum_{n=1}^{\infty}(-1)^n \frac{n^2\beta^2}{4\tau} e^{-n^2\beta^2/4\tau} e^{-\tau X}$$

$$= -\frac{1}{32\pi^2}\int d^4x \sqrt{|g(x)|}\frac{1}{6}R_0^0\sum_{n=1}^{\infty}(-1)^n n^2\beta^2 \int_0^\infty \frac{d\tau}{\tau^3} e^{-(\tau X + n^2\beta^2/4\tau)}$$

$$= -\frac{1}{32\pi^2}\int d^4x\sqrt{|g(x)|}\frac{1}{6}R_0^0\sum_{n=1}^{\infty}(-1)^n n^2\beta^2 2\left(\frac{4X}{n^2\beta^2}\right)k_2(\sqrt{X}n\beta)$$

$$= -\int d^4x \sqrt{|g(x)|}R_0^0\left\{\frac{1}{12\pi^2}Y_f(2)\zeta(2)\beta^{-2} - \frac{1}{96\pi^2}X + \cdots\right\}. \tag{4.5}$$

It is easy to show that, for metric (3.1)

$$\nabla_i\nabla^i(\ln g_{00}) \equiv (\ln g_{00})_{;i}{}^{;i} = -2R_0^0.$$

Comparing (4.4) and (4.5) with (2.22) of [14], we see that:

(i) The three main terms

$$\frac{1}{\pi^2}Y_f(4)\zeta(4)\beta^{-4} - \frac{1}{4\pi^2}Y_f(2)\zeta(2)\beta^{-2}\left(X - \frac{1}{6}R\right)$$

are the same, if we set $X = m^2$ and replace $Y_f(s)$ by $Y_b(s)$.

(ii) $(1/96\pi^2)XR^0_0$ is equal to $-(m^2/192\pi^2)\nabla_i\nabla^i(\ln g_{00})$ (we think that the authors of [14] have missed the first term of (4.5)).

The energy–momentum tensor of the fermion field at finite temperature in four dimensions in a one-loop approximation is

$$T^{(f)\alpha\beta} = \frac{2}{\sqrt{|g(x)|}}\frac{\delta}{\delta g_{\alpha\beta}}W = \frac{2}{\sqrt{|g(x)|}}\frac{\delta}{\delta g_{\alpha\beta}}\ln Z^{(f)}_{\beta_0}. \tag{4.6}$$

Substituting (4.2) into (4.6) the energy–momentum tensor corresponding to the heat kernel expansion of type I is given by

$$T^{(f)00} = -Y_f(4)\frac{3}{\pi^2}g^{00}\zeta(4)\beta^{-4} - Y_f(2)\frac{1}{4\pi^2}\beta^{-2}\left[g^{00}\left(\frac{1}{6}R - X\right) + \frac{1}{3}R^{00}\right] + g^{00}[\gamma + 2\ln 2]E_4(x) + \cdots \tag{4.7}$$

and

$$T^{(f)ij} = Y_f(4)\frac{1}{\pi^2}g^{ij}\zeta(4)\beta^{-4} + Y_f(2)\frac{1}{4\pi^2}\beta^{-2}\left[g^{ij}\left(\frac{1}{6}R - X\right) - \frac{1}{3}R^{ij}\right] + g^{ij}[\gamma + 2\ln 2]E_4(x) + \cdots. \tag{4.8}$$

Substituting (4.4) into (4.6), we obtain the energy–momentum tensor corresponding to heat kernel expansion type II:

$$\begin{aligned}
T^{(f)00} &= \frac{2}{\sqrt{|g(x)|}}\frac{\delta}{\delta g_{00}}\int \mathrm{d}^4x\,\sqrt{|g(x)|}\Bigg[\!\!\Bigg[\Bigg\{\frac{1}{\pi^2}Y_f(4)\zeta(4)\beta^{-4} \\
&\quad + \frac{1}{4\pi^2}Y_f(2)\zeta(2)\beta^{-2}\left(\frac{1}{6}R - X\right)\Bigg\} + \frac{1}{96\pi^2}\Bigg\{\left[2\gamma - 1 + \ln\left(\frac{1}{4}X\beta^2\right)\right]XR \\
&\quad - 3\left[\gamma - \frac{3}{4} + \frac{1}{2}\ln\left(\frac{1}{4}X\beta^2\right)\right]X^2\Bigg\} + \cdots\Bigg]\!\!\Bigg] \\
&= g^{00}\Bigg\{ - Y_f(4)\frac{3}{\pi^2}\zeta(4)\beta^{-4} - Y_f(2)\frac{1}{4\pi^2}\zeta(2)\beta^{-2}\left(\frac{1}{6}R - X\right) \\
&\quad + \frac{1}{96\pi^2}\left[2\gamma - 1 + \ln\left(\frac{1}{4}X\beta^2\right)\right]XR - \frac{1}{32\pi^2}\left[\gamma - \frac{3}{4} + \frac{1}{2}\ln\left(\frac{1}{4}X\beta^2\right)\right]X^2\Bigg\} \\
&\quad - Y_f(2)\frac{1}{12\pi^2}\zeta(2)\beta^{-2}R^{00} + \frac{1}{48\pi^2}\Bigg[- \left(2\gamma - 1 + \ln\frac{1}{4}X\beta^2\right)XR^{00} \\
&\quad + g^{00}\nabla^2(X\ln X) - \nabla^0\nabla^0(X\ln X)\Bigg]
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
T^{(f)ij} &= g^{ij}\Bigg\{\frac{1}{\pi^2}Y_f(4)\zeta(4)\beta^{-4} + \frac{1}{4\pi^2}Y_f(2)\zeta(2)\beta^{-2}\left(\frac{1}{6}R - X\right) \\
&\quad + \frac{1}{96\pi^2}\left[2\gamma - 1 + \ln\frac{1}{4}X\beta^2\right]XR - \frac{1}{32\pi^2}\left[\gamma - \frac{3}{4} + \frac{1}{2}\ln\left(\frac{1}{4}X\beta^2\right)\right]X^2\Bigg\} \\
&\quad - \frac{1}{12\pi^2}Y_f(2)\zeta(2)\beta^{-2}R^{ij} + \frac{1}{48\pi^2}\Bigg[- \left(2\gamma - 1 + \ln\frac{1}{4}X\beta^2\right)XR^{ij} \\
&\quad + g^{ij}\nabla^2(X\ln X) - \nabla^i\nabla^j(X\ln X)\Bigg].
\end{aligned} \tag{4.10}$$

The contribution of the anomalous term to the energy–momentum tensor is

$$T_a^{(f)00} = -3\left[\frac{1}{12\pi^2}Y_f(2)\zeta(2)\beta^{-2} - \frac{1}{96\pi^2}X + \ldots\right]R^{00} + \frac{1}{6\pi^2}g^{00}Y_f(2)\zeta(2)\beta^{-2}R_0^0$$

$$= -\frac{1}{12\pi^2}Y_f(2)\zeta(2)\beta^{-2}R^{00} + \frac{1}{32\pi^2}XR^{00} + \cdots \quad (4.11)$$

and

$$T_a^{(f)ij} = -g^{ij}R_0^0\left[\frac{1}{12\pi^2}Y_f(2)\zeta(2)\beta^{-2} - \frac{1}{96\pi^2}X + \cdots\right]. \quad (4.12)$$

To simplify the calculation we have assumed that X is independent of the metric $g_{\alpha\beta}$ in deriving (4.8) to (4.12).

5. Summary and discussion

We have derived the heat kernel expansion coefficients $E_m^{(f)}(x, \beta_0, \tau)$ and $F_m^{(f)}(x, \beta_0, \tau)$, $m = 0, 2, 4$, the effective action and the energy–momentum tensor for a fermion field at finite temperature in the general static curved spacetime. The results (4.4), (4.9) and (4.10) essentially agree with (2.20) and (3.1*a*, *b*) of [4] for a boson field if $Y_f(s)$ is replaced by $Y_b(s)$ in the corresponding equations. However, there exists a little difference, since we assume that X does not involve ξR.

The apparently different forms of (4.2) and (4.4) and the corresponding energy–momentum tensors are not really different, they are merely different resummations.

The algorithm used here does not utilize the Riemannian normal coordinates (RNC). Equation (2.8) plays the same role in momentum space representation as RNC do in the usual spacetime representation.

The role of the parameter τ is similar to the parameter is of [1, 2], yet we use the Euler transformation instead of the Mellin transformation.

X may involve or be equal to the m^2 term but we did not introduce the operator $\partial/\partial X$.

The heat kernel expansion is, in fact, a geometric regularization technique.

Each one of $E_2^{(f)}(x, \beta_0, \tau)$, $E_2^{(b)}(x, \beta_0, \tau)$, $F_2^{(f)}(x, \beta_0, \tau)$ and $F_2^{(b)}(x, \beta_0, \tau)$ has their own anomalous term. It is a universal property that the heat kernel expansion coefficients at finite temperature in curved spacetime contain anomalous terms, which need to be studied further.

The algorithm used may be directly extended to spacetime with torsion, and to higher-order differential operators, for instance, a fourth-order differential operator, which is related to quantum gravity [16, 17].

Acknowledgments

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Appendix A

From (3.15) the thermal DeWitt–Seeley–Gilkey coefficient $E_2^{(f)}(x,\beta_0,\tau)$ is given by

$$
\begin{aligned}
E_2^{(f)}(x,\beta_0,\tau) &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}}\int_c \frac{\mathrm{i}\,\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\lambda} \\
&\quad\times \sum_{n=-\infty}^{\infty}\frac{2\pi}{\beta_0}\delta\left[\frac{k_0}{\sqrt{\tau}}-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\right][\![\sigma_2]\!](x,\beta_0,\tau) \\
&= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}}\int_c \frac{\mathrm{i}\,\mathrm{d}\lambda}{2\pi}\mathrm{e}^{-\lambda}\sum_{n=-\infty}^{\infty}\frac{2\pi\sqrt{\tau}}{\beta_0}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\sqrt{\tau}\right] \\
&\quad\times\left[\frac{2}{3}B^3 k_\alpha k_\beta R^{\alpha\beta}-B^2 X\right] \\
&= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{|g(x)|}}\mathrm{e}^{-k^2}\frac{2\pi\sqrt{\tau}}{\beta_0}\sum_{n=-\infty}^{\infty}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\sqrt{\tau}\right] \\
&\quad\times\left(\frac{1}{3}k_\alpha k_\beta R^{\alpha\beta}-X\right).
\end{aligned}
\tag{A1}
$$

If the spacetime is static and the metric may be written as (3.1), the parameter k_α can be decomposed into $k_\alpha = \{k_0, \hat{k}_j\}$, where $\hat{k}_j$ does not contain k_0 component

$$
\begin{aligned}
E_2^{(f)}(x,\beta_0,\tau) &= \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^{(d-1)}\hat{k}}{\sqrt{|\hat{g}(x)|}}\int \frac{\mathrm{d}k_0}{\sqrt{g_{00}(x)}}\mathrm{e}^{-\hat{k}_j\hat{k}^j}\mathrm{e}^{-g^{00}k_0k_0}\frac{2\pi\sqrt{\tau}}{\beta_0} \\
&\quad\times\sum_{n=-\infty}^{\infty}\delta\left[k_0-\frac{2\pi}{\beta_0}\left(n+\frac{1}{2}\right)\sqrt{\tau}\right]\left(\frac{1}{3}k_0k_0R^{00}+\frac{1}{3}k_ik_jR^{ij}-X\right).
\end{aligned}
\tag{A2}
$$

Using the integration formula

$$
I(n,s) = \frac{1}{(2\pi)^n}\int \frac{\mathrm{d}^n k}{\sqrt{|g(x)|}}k_{\alpha_1}k_{\alpha_2}\ldots k_{\alpha_{2s}}\mathrm{e}^{-k^2} = \frac{1}{(4\pi)^{1/2n}2^s}g_{\{\alpha_1\alpha_2\ldots\alpha_{2s}\}} \tag{A3}
$$

where $g_{\{\alpha_1\alpha_2\ldots\alpha_{2s}\}}$ is the symmetrized combination of metric tensors (it involves $(2s-1)!!$ terms), for instance,

$$
g_{\{\alpha\beta\lambda\tau\}} = g_{\alpha\beta}g_{\lambda\tau}+g_{\alpha\lambda}g_{\beta\tau}+g_{\alpha\tau}g_{\beta\lambda} \tag{A4}
$$

and the generalized theta function transformation [7]

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\left[a\left(n+\frac{1}{2}\right)^2\right]^p \mathrm{e}^{-a(n+1/2)^2} \\
&\quad = \left(\frac{\pi}{a}\right)^{1/2}\frac{(2p-1)!!}{2^p}\left\{1+2\sum_{n=1}^{\infty}(-1)^n\mathrm{e}^{-n^2\beta^2/a}\sum_{l=1}^{p-1}\left(-\frac{\pi^2n^2}{a}\right)^l C_l^p\right\}
\end{aligned}
\tag{A5}
$$

where

$$C_l^p = \frac{2^l p!}{l!(p-l)!(2l-1)!!}$$

finally, we obtain

$$E_2^{(f)}(x,\beta_0,\tau) = \frac{1}{(4\pi)^{1/2d}}\left\{\left(\frac{1}{6}R - X\right)\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\beta^2/4\tau}\right]\right.$$
$$\left. - \frac{2}{3}R_0^0\sum_{n=1}^{\infty}(-1)^n\frac{n^2\beta^2}{4\tau}e^{-n^2\beta^2/4\tau}\right\} \tag{A6}$$

where $\beta = \sqrt{g_{00}}\beta_0$ and $\beta_0 = \text{constant}$.

Appendix B

The coincidence limit of five time covariant derivatives of $l(x,x',k)$ is

$$\begin{aligned}
[\![\nabla_{\alpha\beta\lambda\tau\sigma} l(x,x',k)]\!] &\equiv [\![\nabla_{\alpha\beta\lambda\sigma\tau} l(x,x',k)]\!] \equiv k_\rho l_{\alpha\beta\lambda\tau\sigma}{}^\rho \\
= &- \tfrac{1}{30}k_p\{4(4\nabla_{\alpha\beta}+\nabla_{\beta\alpha})S^p{}_{\tau\lambda\sigma} + (\nabla_{\alpha\lambda}+4\nabla_{\lambda\alpha})S^p{}_{\tau\beta\sigma} + (\nabla_{\alpha\tau}+4\nabla_{\tau\alpha})S^p{}_{\lambda\beta\sigma} \\
&+ (\nabla_{\alpha\sigma}+4\nabla_{\sigma\alpha})S^p{}_{\lambda\beta\tau} + (\nabla_{\beta\lambda}+\nabla_{\lambda\beta})S^p{}_{\tau\alpha\sigma} + (\nabla_{\beta\tau}+\nabla_{\tau\beta})S^p{}_{\lambda\alpha\sigma} \\
&+ (\nabla_{\beta\sigma}+\nabla_{\sigma\beta})S^p{}_{\lambda\alpha\tau} + (\nabla_{\lambda\tau}+\nabla_{\tau\lambda})S^p{}_{\beta\alpha\sigma} + (\nabla_{\lambda\sigma}+\nabla_{\sigma\lambda})S^p{}_{\beta\alpha\tau} \\
&+ (\nabla_{\tau\sigma}+\nabla_{\sigma\tau})S^p{}_{\beta\alpha\lambda}\} + \tfrac{4}{15}k_\rho\{S^\omega{}_{\lambda\alpha\tau}S^p{}_{\omega\beta\tau} + S^\omega{}_{\tau\alpha\sigma}S^p{}_{\omega\beta\lambda} + S^\omega{}_{\lambda\beta\sigma}S^p{}_{\omega\alpha\tau} \\
&+ S^\omega{}_{\tau\beta\sigma}S^p{}_{\omega\alpha\lambda} + S^\omega{}_{\tau\lambda\sigma}S^p{}_{\omega\alpha\sigma} + S^\omega{}_{\lambda\beta\tau}S^p{}_{\omega\alpha\sigma} + S^\omega{}_{\lambda\alpha\tau}S^p{}_{\omega\beta\sigma} + S^\omega{}_{\beta\alpha\lambda}S^p{}_{\sigma\omega\tau}\} \\
&+ \tfrac{2}{15}k_p\{R^\omega{}_{\beta\alpha\sigma}S^p{}_{\tau\omega\lambda} + R^\omega{}_{\beta\alpha\tau}S^p{}_{\sigma\omega\lambda} + R^\omega{}_{\tau\alpha\beta}S^p{}_{\omega\lambda\sigma} + R^\omega{}_{\sigma\alpha\beta}S^p{}_{\omega\lambda\tau}\} \\
&- \tfrac{4}{15}k_p\{S^\omega{}_{\beta\alpha\tau}S^p{}_{\lambda\omega\sigma} + S^\omega{}_{\lambda\alpha\tau}S^p{}_{\beta\omega\sigma} + S^\omega{}_{\lambda\alpha\sigma}S^p{}_{\beta\omega\tau} + S^\omega{}_{\lambda\beta\sigma}S^p{}_{\alpha\omega\tau} + S^\omega{}_{\tau\alpha\sigma}S^p{}_{\beta\omega\lambda} \\
&+ S^\omega{}_{\lambda\beta\tau}S^p{}_{\alpha\omega\sigma} + S^\omega{}_{\tau\lambda\sigma}S^p{}_{\alpha\omega\beta} + S^\omega{}_{\tau\beta\sigma}S^p{}_{\alpha\omega\lambda} \\
&+ S^\omega{}_{\beta\alpha\sigma}S^p{}_{\lambda\omega\tau} + S^\omega{}_{\beta\alpha\lambda}S^p{}_{\tau\omega\sigma}\}.
\end{aligned} \tag{B1}$$

The coefficients of equation (B1) are more regular than equation (9) of [6]. From (B1) we obtain

$$\begin{aligned}
l_\alpha{}^\alpha{}_\beta{}^\beta{}_\lambda{}^\lambda &= \tfrac{2}{5}\Box R - \tfrac{1}{15}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{4}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} \\
l_{\alpha\beta}{}^\alpha{}_\lambda{}^{\lambda\beta} &= -\tfrac{3}{5}\Box R - \tfrac{2}{5}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{4}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} \\
l_{\alpha\beta}{}^{\beta\alpha}{}_\lambda{}^\lambda &= \tfrac{2}{5}\Box R + \tfrac{4}{15}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{11}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} \\
l_{\alpha\beta}{}^\beta{}_\lambda{}^{\lambda\alpha} &= -\tfrac{3}{5}\Box R + \tfrac{4}{15}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{4}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} \\
l_{\alpha\beta\lambda}{}^{\lambda\alpha\beta} &= -\tfrac{1}{10}\Box R - \tfrac{1}{15}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{4}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau} \\
l_{\alpha\beta\lambda}{}^{\lambda\beta\alpha} &= -\tfrac{1}{10}\Box R - \tfrac{2}{5}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{4}{15}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau}.
\end{aligned} \tag{B2}$$

Appendix C

One type (type I) of heat kernel expansion for fermion at finite temperature is

$$H^{(f)}(x,\beta_0,\tau)=\sum_{l=0}^{\infty}H_{2l}^{(f)}(x,\beta_0,\tau)=\sum_{l=0}^{\infty}\tau^{(2l-d)/2}E_{2l}^{(f)}(x,\beta_0,\tau) \tag{C1}$$

where

$$E_0^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\theta_4(z|\hat{\tau}) \tag{C2}$$

$$E_2^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\left(\frac{1}{6}R-X\right)\theta_4(z|\hat{\tau})+A_2^{(f)}(x,\beta_0,\tau) \tag{C3}$$

$$\begin{aligned}E_4^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\bigg\{&\frac{1}{180}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau}-\frac{1}{180}R_{\alpha\beta}R^{\alpha\beta}+\frac{1}{72}R^2+\frac{1}{30}\Box R\\&-\frac{1}{6}XR+\frac{1}{2}X^2-\frac{1}{6}\Box X+\frac{1}{12}W_{\alpha\beta}W^{\alpha\beta}\bigg\}\theta_4(z|\hat{\tau})+A_4^{(f)}(x,\beta_0,\tau)\end{aligned} \tag{C4}$$

$$A_2^{(f)}(x,\beta_0,\tau)=-\frac{1}{(4\pi)^{d/2}}\frac{2}{3}R_0^0\sum_{n=1}^{\infty}(-1)^n\frac{n^2\beta^2}{4\tau}e^{-n^2\beta^2/4\tau}. \tag{C5}$$

The other type (type II) is

$$H^{(f)}(x,\beta_0,\tau)=\sum_{l=0}^{\infty}H_{2l}^{(f)}(x,\beta_0,\tau)=\sum_{l=0}^{\infty}\tau^{(2l-d)/2}e^{-\tau X}F_{2l}^{(f)}(x,\beta_0,\tau) \tag{C6}$$

where

$$F_0^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\theta_4(z|\hat{\tau}) \tag{C7}$$

$$F_2^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\frac{1}{6}R\theta_4(z|\hat{\tau})+A_2^{\prime(f)}(x,\beta_0,\tau) \tag{C8}$$

$$\begin{aligned}F_4^{(f)}(x,\beta_0,\tau)=\frac{1}{(4\pi)^{d/2}}\bigg\{&\frac{1}{180}R_{\alpha\beta\lambda\tau}R^{\alpha\beta\lambda\tau}-\frac{1}{180}R_{\alpha\beta}R^{\alpha\beta}+\frac{1}{72}R^2+\frac{1}{30}\Box R\\&-\frac{1}{6}\Box X+\frac{1}{12}W_{\alpha\beta}W^{\alpha\beta}\bigg\}\theta_4(z|\hat{\tau})+A_4^{\prime(f)}(x,\beta_0,\tau)\end{aligned} \tag{C9}$$

$$A_2^{\prime(f)}(x,\beta_0,\tau)=A_2^{(f)}(x,\beta_0,\tau). \tag{C10}$$

We have derived (C6)–(C10) by a direct method of operator formalism utilizing the Baker–Campbell–Hausdorff formula [20]. On the other hand, it is not difficult to derive (C1)–(C5) from (C6)–(C10) by expanding $e^{-\tau X}$ in a power series, and rearranging or recollecting terms according to the powers of τ.

References

[1] Dewitt B S 1965 *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)
[2] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[3] Christensen S M (ed) 1984 *Quantum Theory of Gravity* (Bristol: Adam Hilger)
[4] Seeley R T 1967 *Proc. Symp. on Pure Mathematics (American Mathematical Society 10)* p 288
[5] Gilkey P B 1975 *J. Diff. Geom.* **10** 601; 1979 *Compos. Math.* **38** 201
[6] Gusynin V P 1989 *Phys. Lett.* **225B** 233
[7] Boschi-Filho H and Natividade C P 1992 *Phys. Rev.* D **45** 586
[8] Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320
[9] Dowker J S and Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** 895
[10] Braden H W 1982 *Phys. Rev.* D **25** 1028
[11] Hu B L 1983 *Phys. Lett.* **123B** 189
[12] Weldon H A 1982 *Phys. Rev.* D **26** 1399
[13] Hu B L, Critchley R and Stylianopoulos A 1987 *Phys. Rev.* D **35** 510
[14] Nakazawa N and Fukuyama T 1985 *Nucl. Phys.* B **252** 621
[15] Nakazawa N 1985 *Prog. Theor. Phys.* **74** 375
[16] Julve J and Tonin M 1978 *Nuovo Cimento* B **46** 137
[17] Stelle K 1977 *Phys. Rev.* D **16** 953
[18] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals Series and Products* (London: Academic) p 959 8.432(6) p 961 8.446
[19] Erdelyi A and Bateman H 1954 *Tables of Integral Transformations* vol 1 (New York: McGraw-Hill) p 146 (29)
[20] Xu Dian-Yan in preparation